The rise of an isolated thermal through stratified surroundings

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The problem being considered here is the behaviour of a dry thermal in various stable density gradients, although the calculated examples have been formulated in another way: that is, gradients have been found which allow the thermal to behave in certain specified ways.

It has been found, using model experiments, that a dry thermal rising in neutral surroundings will have similar motion at all stages and will hence grow along a cone. This growth is made possible by the entrainment of outside fluid which takes place by mixing through the front and at the rear in a smooth upward "jet". About 40% of the total entrained volume can be estimated from the experiments to come in at the rear and the remaining 60% at the front.

Fig. 1 Thermal in neutral surroundings

Consider now an atmosphere which is slightly stable. Then the behaviour of the thermal in it will only be slightly different from its behaviour in neutral surroundings and it is supposed that this difference will only be in the amount of fluid swept up from lower levels, which has negative buoyancy. Mixing at the back will thus be inhibited and less fluid will enter into the "jet". Mixing at the front will still occur and is supposed not to be affected by the small difference in density from the neutral gradient.

The motion of the thermal in a gradient which is more stable than before will show the same trends; less fluid being entrained at the back and the same amount being entrained at the front. It can be seen that eventually a gradient will be reached such that no entrainment takes place at the back and the thermal grows only through the mixing on its front surface. There will now be a dividing point in the flow which is on the surface of the thermal (see fig. 2), although the inside of the thermal will still have the vortex ring type of circulation, but this will be weaker than that of a thermal rising in neutral surroundings.

Now, with a further increase in the stability of the gradient a thermal rising would leave behind a wake of fluid which had been eroded away from it.

Theory

In finding the following formula, in each case the basis has been the equations of conservation of mass and volume, an assumption about the density of fluid entrained at the front, and the use of a formula expressing the similarity at all stages of the motion, i.e.

$C'_n \frac{d\rho_0}{dt} = \rho_1 - \rho_0$, $n = 1, 2, 3, 4$, one for each of the following four cases

where,
C'_n = similarity constant
$\frac{d}{dt}$ = differential operator with respect to time
ρ_0 = density of the surroundings
ρ_1 = average density of the thermal

Densities are used here for convenience in relating these formulae to experiment, and are, of course, equivalent to potential temperatures in the atmosphere.

The first situation considered is the critical one where no fluid is being entrained at the rear but where there is no wake. It can be shown that the thermal grows along a cone and that the necessary gradients of density are solutions of the hypergeometric equation,

$$Z\rho''_0 + (m - Z)\,\rho'_0 + n\,\rho_0 = 0$$

i.e.

$$\rho_0 = K_1\,{}_1F_1(-n, m; Z) + K_2 Z^{1-m}\,{}_1F_1(1 - n - m, 2 - n; Z)$$

where primes on ρ_0 denote differentiation with respect to Z, and

Z = distance of the top of the thermal from the virtual origin
$m = 1/C'_2$
$n = 3(1 - q)/C'_2$
q = proportion of the fluid which is entrained at the front of a thermal rising in neutral surroundings.
${}_1F_1$ = symbol for the confluent hypergeometric function of the first kind. (See ref. [1] and other texts on Analysis or Differential Equations).

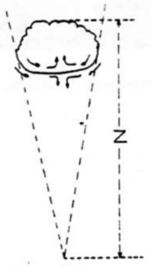

Fig. 2 Thermal with no wake

In this solution there are two arbitrary constants K_1, K_2 one of which will be specified by the starting conditions.

For the regime where there is no wake but some entrainment at the back, a density gradient has been calculated which will allow the thermal to grow along a cone. This is,

$$\rho_0 = K_1 C_1'^{-2} \int_{Z_0}^{Z} u^{-3}\, e^{-u}\, du + \rho_{00}$$

Fig. 3 Thermal in weak density gradient

Here ρ_{00} is the density of fluid at height Z_0 and u is the dummy symbol used in writing down the integral.

Two particular cases have been worked out for the regime in which there is a wake. Firstly the case where the volume of fluid being mixed in at the front is equal to that being left behind in the wake; the thermal thus remaining the same size. The required density here is of the form,

$$\rho_0 = K_1 e^{pz} + K_2$$

where p is the solution of the equation,

$$p^2 - \frac{1}{C'_3}\,p + \frac{\Pi}{MdC'_3}\,(1 - e^{-hp}) = 0$$

and,
d = diameter of wake
h = distance from the top of the termal to the level where the wake is in equilibrium with the surroundings
M = volume constant ($V = Md_3$)

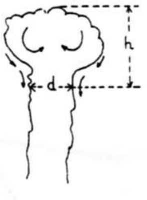

Fig. 4 Thermal of constant size

In the second case the motion is specified to be along a decreasing cone and the required gradient is found to be the solution of the equation,

$$C'_4 Z \rho''_0(Z) + (Z + 3C'_4)\,\rho'_0(Z) = 3(A - 1)\,\rho_0(Z) - 3(A + 1)\,\rho_0(pZ)$$

where, $A = \frac{Mq}{N^3} \cdot \frac{N'^3}{M'}$

$p = 1 - k/N'$
M, N = constants appearing in the motion of a thermal in neutral surroundings
$N' = \cot\gamma$
M' = volume constant
$2r$ = maximum diameter of the thermal
kr = distance from the top of the thermal to the level where the wake is in equilibrium with the surroundings

The solution of the equation can be found by Frobenius' method to be the sum of two series in increasing powers of Z, i.e.,

$$\rho_0 = K_1 S_1(Z) + K_2 S_2(Z).$$

Thus, if the parameters appearing in these solutions were known, and experiments are going on at Imperial College which might be modified to find them, it would be possible to draw a graph something like the following for density (or potential temperature) against height, illustrating the regimes of different thermal behaviour.

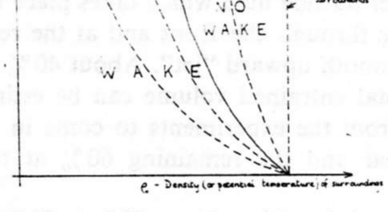

Fig. 6 Graph of different regimes of thermal behaviour

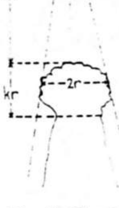

Fig. 5 Thermal decreasing along a cone

References

[1] Jeffreys, Sir H., 1956, "Methods of Mathematical Physics", Cambridge U. Pr.

[2] Scorer, R.S. and Ronne, C., 1956, "Weather", 11, p. 151

[3] Woodward, B., 1959, "Quart. J.R.Met.Soc.", 85, p. 144