Wind flow over small hills

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Summary: The paper deals with a modification of Rankine's method of modelling the flow of a perfect fluid, based on the use of a system-flow and source as modelling element for the stationary flow. The isogones and the isotachs of such a basic unit are circles. Drawing these circles, one can realize the flow over a symmetrical hill, finding the streamlines by the method of tangents. A generalization of the theorem on maximum vertical velocity over a hill is also proved.

1. Basic equations: Wind flow over small hills, of the height of 50 to 100 metres, can be examined by the methods of hydrodynamics of perfect fluids. The picture of the air flow thus obtained has its physical meaning only on the windward side of the hill, since on the other side rotors occur.

Rankine's method, replacing the overflown rigid body by a system of sinks and sources, furnishes a splendid mathematical tool. The superposition on such a velocity field of an air stream with a constant velocity parallel to the horizontal axis creates a kinematic system, whose streamlines are the trajectories of fluid particles and in limiting cases present the shape of the overflown body.

Rankine's method takes as the basic element a source or a sink, defined in what follows by the formula

$$w = \frac{m}{2\pi} \ln z \qquad (1.1)$$

where $\frac{m}{2\pi}$ is the efficiency of the source, $z = x + iy$ a complex number.

Instead of using the properties of this function separately, in what follows we take as the basic system the following group: airstream and source, defined by the equation

$$w = Uz + \frac{m}{2\pi} \ln z \qquad (1.2)^*$$

where U is the wind velocity. This function in meteorology describes the air flow over a steep hill (Steilküste).

From the well known hydrodynamic relationships one can write for the stream function and for the velocity potential

$$\Psi = Ur \cdot \sin \vartheta + \frac{m}{2\pi} \vartheta \qquad (1.3)$$

and $\Phi = Ux + \frac{m}{2\pi} \ln r \qquad (1.4)$

where r and ϑ are polar coordinates.

Differentiating (1.2) with respect to z, we get

$$\frac{dw}{dz} = U + \frac{m}{2\pi z} \qquad (1.5)$$

The horizontal and vertical velocity components, v_x and v_y, are then

$$v_x = U + \frac{m}{2\pi} \frac{x}{r^2} \qquad (1.6)$$

* It is to remember that $\frac{dw}{dz} = V_x - i \cdot V_y$

$$v_y = \frac{m}{2\pi} \frac{y}{r^2} \qquad (1.7)$$

It is to be remarked here, that the vertical wind component in equation (1.1) and (1.2) is the same.

With these formulae, one can compute the equations of isogones and isotachs for the system (1.2). The isogones are defined by the equation

$$\operatorname{tg} \Phi = C = \frac{v_y}{v_x} = \frac{\frac{m}{2\pi} \frac{y}{r^2}}{U + \frac{m}{2\pi} \frac{x}{r^2}}$$

which is the equation of a circle.

$$\left(x + \frac{m}{4\pi u}\right)^2 + \left(y - \frac{m}{4\pi CU}\right)^2 = \frac{m^2}{16\pi^2 U^2}\left(1 + \frac{1}{C^2}\right) \qquad (1.8)$$

This circle has the following properties:

1. it passes through the coordinates origine (source)
2. its radius is $\frac{m}{4\pi U \sin\Phi}$, and its center lies in the point $\left(-\frac{m}{4\pi u} \,\middle|\, \frac{m}{4\pi c \cdot u}\right)$
3. the centers of successive circles of the family of isotachs lie on a straight line through the origin, inclined to the axis OX by the angle $\frac{\pi}{2} + \Phi$, where Φ is the isogone angle.

The zero isogone is reduced to the OX axis, and the isogone of the perpendicular directions is found from the relation

$$V_x = U + \frac{mx}{2\pi r^2} = O \qquad (1.9)$$

This is the equation of a circle through the coordinates origine and tangent to the OX axis. The shape can be found from the following form

$$\left(x + \frac{m}{4\pi U}\right)^2 + y^2 = \frac{m^2}{16\pi^2 U^2} \qquad (1.10)$$

For $y = 0$, it follows $x = -\frac{m}{4\pi U} \pm \frac{m}{4\pi U}$ wherefrom

$$x_1 = O,\ x_2 = -\frac{m}{2\pi U} \qquad (1.11)$$

The point with the coordinates $(x_2, 0)$ is the critical point, in which the flow velocity is zero.

In a similar manner as before, we find the equation of the isotach. Then $v^2 = c^2 = V_x^2 + V_y^2 = U^2 + \frac{m x U}{\pi r^2} + \frac{m^2}{4\pi^2 r^2}$ or

$$\left(x - \frac{m U}{2\pi (c^2 - U^2)}\right)^2 + y^2 = \frac{m^2 c^2}{4\pi^2 (C^2 - U^2)^2} \qquad (1.12)$$

It is a circle with the center in the point $\left(\frac{. \, m \, U}{2\pi\,(c^2 - U^2)}, O\right)$

and radius $R = \frac{m \, c}{2\pi\,(c^2 - U^2)}$

The case c = U, or of the velocity equal to coming airflow needs a special consideration. The equation (1.1) is reduced to a straight line, defined by the equation

$$U_x + \frac{m}{4\pi} = O \qquad (1.13)$$

The wind velocity plays in a certain sense the role of the critical velocity, which divides by means of this straight line the space around the hill, into two parts, one with the wind velocity below the undisturbed flow velocity, and another with the wind velocity surpassing the undisturbed flow velocity.

This study of the kinematic properties of the system-source and parallel airstream proves that its isogones and isotachs form circles, which are lines easy to draw.

2. A symmetrical hill: A symmetrical hill is composed by two systems-source and sink of equal efficiency, under the action of the airstream parallel to the OX axis. In mathematical language, it is defined by the equation

$$w = Uz + \frac{m}{2\pi} \ln\left(\frac{z+a}{z-a}\right) \qquad (2.1)$$

where 2a is the distance between the source and sink. This function may also be written in the following form

$$w = \frac{Uz}{2} + \frac{m}{2\pi} \ln(z+a) + \frac{Uz}{2} - \frac{m}{2\pi} \ln(z-a) \qquad (2.2)$$

Here the flow function is the sum of the flow functions of the two systems-airflow and source, and airflow and sink, the kinematic properties of which were studied. The isogones and the isotachs of the kinematic system-symmetrical hill are then found by the well known Maxwell method, that is by summing up the values of respective isolines in their point of intersection and thus drawing the resultant isoline.

It is to be pointed out that the vertical components of the wind (Aufwind) remain here unchanged, since the incoming airstream, with its velocity parallel to the horizontal axis of the hill's cross-section, does not have any vertical component. The advantage of this method of determining the field of the velocity consists in the fact that the component elements are relatively simple curves, circles, which, due to the symmetry of the field with respect to the sink, are the same in both parts of the whole system. The isogones, which form the base to draw the stream-lines by the tangent method, have a physical meaning on the windward side of the hill.

3. The problem of extreme winds over a steep hill: The vertical wind component along the same stream-line changes its value, being small at great distances from the hill and growing to a maximum in the hill's neighbourhood. This problem had been solved by Glauert for zero stream-line. Now we treat here its generalization for any stream-line.

According to (1.7), the vertical wind component is defined by

$$V_y = \frac{my}{2\pi r^2} \qquad (3.1)$$

Along the same stream-line, it will be (1.3)

$$r = \frac{\Psi - \frac{m}{2\pi}\vartheta}{U \sin\vartheta} \qquad (3.2)$$

The vertical component of the stream (3.1) along the same stream-line will be

$$V_y = \frac{m}{2\pi r} \cdot \frac{y}{r} = \frac{m}{2\pi r} \cdot \sin\vartheta = \frac{m \, U \sin^2\vartheta}{2\pi\left(\Psi - \frac{m}{2\pi}\vartheta\right)} \qquad (3.3)$$

This equation, differentiated with respect to ϑ and equated to zero, yields a condition which allows us to compute these values of ϑ, that correspond to extremes of the wind, in our case to the maximum. In this manner arises the following equation

$$\operatorname{tg}\vartheta = \frac{4\pi}{m}\left(\frac{m\vartheta}{2\pi} - \Psi\right) \qquad (3.4)$$

If $\Psi = 0$, the e quations (3.3) and (3.4) reduce to equations, obtained by Glauert, and after him, by Sutton.

Eliminating from (3.4) the magnitude $U_y = \Psi - \frac{m}{2\pi}\vartheta$ by the use of (3.2) we get the locus of the points of maximum vertical wind components. It is the straight line with the equation

$$x = \frac{m}{4\pi U} \qquad (3.5)$$

which is parallel to the OX axis.

The study of extreme wind for horizontal component does not yield such a simple relationship.

Literature

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